The multi-type bisexual Galton-Watson process with superadditive mating

Nicolás Zalduendo

Joint work with Coralie Fritsch and Denis Villemonais

Mathematical Models in Ecology and Evolution
Reading University

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- Later.

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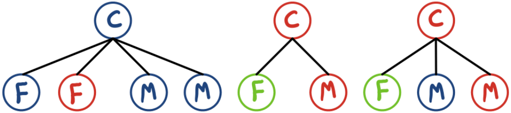

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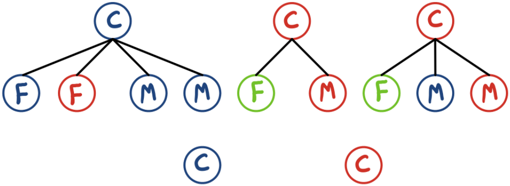

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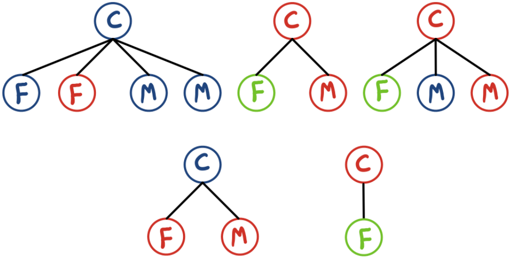

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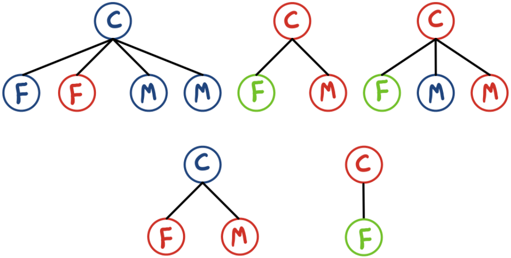

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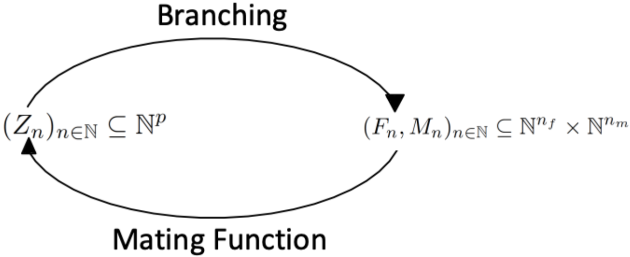
$(Z_n)_{n\in\mathbb{N}}$ is the multi-type bisexual Galton-Watson process with *mating function* ξ.

The multi-type bGW process

In general,



for a mating function $\xi : \mathbb{R}_+^{n_f} \times \mathbb{R}_+^{n_m} \longrightarrow \mathbb{R}_+^{p}$.

Superadditive Mating

We assume that for all $x_1, x_2 \in \mathbb{R}_+^{n_f}$ and for all $y_1, y_2 \in \mathbb{R}_+^{n_m}$,

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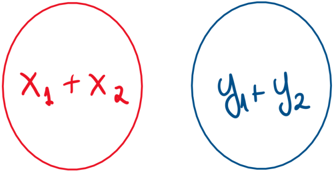

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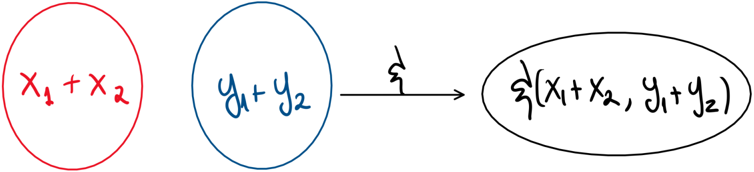

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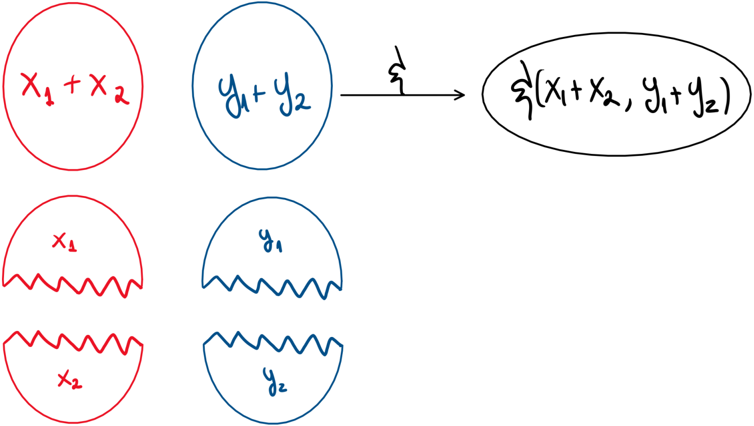

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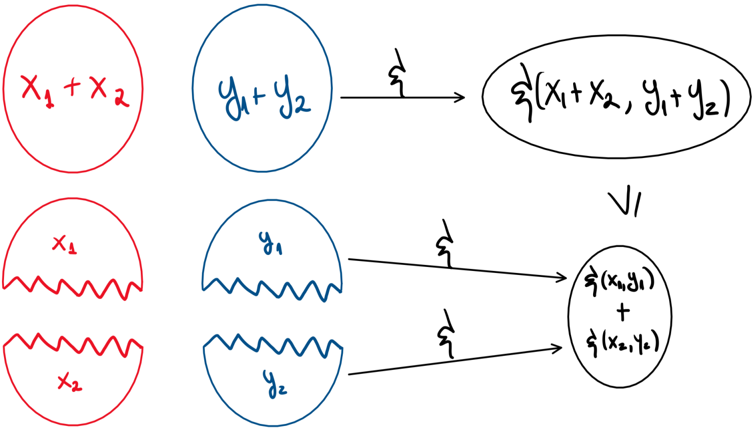


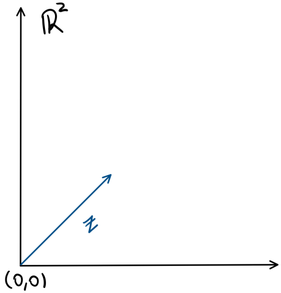
The R Function

Proposition

The function $R : \mathbb{R}_+^p \longrightarrow (\mathbb{R}_+ \cup \{+\infty\})^p$ given by

$$R(z) = \lim_{k \to +\infty} \frac{\mathbb{E}(Z_1 \mid Z_0 = \lfloor kz \rfloor)}{k}$$

is well defined.



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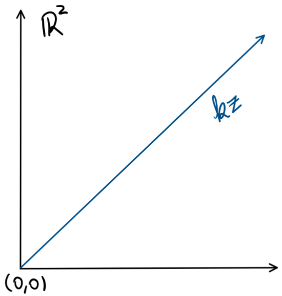

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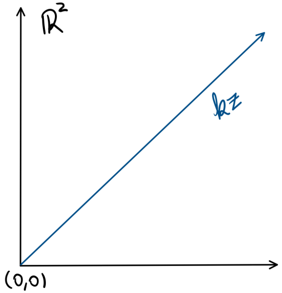

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Theorem [Fritsch - Villemonais - Z.]

Consider $z \in \mathbb{N}^p$ and let $Z_0^{(k)} \sim_{k \to +\infty} kz$.
Then, for all $n \geq 0$,

$$Z_n^{(k)} \sim_{k \to +\infty} R^n(kz) \text{ a.s. and in } L^1.$$

Fact: R is concave.

Condition for certain extinction

For the rest, extra assumptions:

- Transcience, which implies:

$$\mathbb{P}\left(\lim_{n\to\infty} |Z_n| \in \{0, +\infty\} \middle| Z_0 = z\right) = 1, \quad \forall z \in \mathbb{N}^p.$$

- There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $z \neq 0$,

$$R^n(z) > 0$$

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Theorem [Krause, '94]

1. The eigenvalue problem

$$R(z^*)=\lambda^* z^*$$

   has a unique solution with $\lambda^*>0$ and $z^*\in(\mathbb{R}_+)^p$, $z^*>0$, $|z^*|=1$.
2. There exists a function $\mathcal{P}:(\mathbb{R}_+)^p\longrightarrow\mathbb{R}_+$ such that

$$\lim_{n\to+\infty}\frac{R^n(z)}{(\lambda^*)^n}=\mathcal{P}(z)z^*$$

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Theorem [Fritsch - Villemonais - Z.]

Assume R is finite. Then,

$$\lambda^* \leq 1 \iff q_z = 1, \forall z \in \mathbb{N}^p.$$

If $\lambda^* > 1$ or if there exists $z' \in (\mathbb{R}_+)^p$ such that $R(z')$ is not finite, then there exists $r > 0$ such that $q_v < 1$ for all $v \in \mathbb{N}^p$ with $|v| \geq r$.

Asymptotic Behaviour

Set

$$X_{i,j} = \#\text{females of type } j \text{ produced by a single couple of type } i.$$
$$Y_{i,j} = \#\text{males of type } j \text{ produced by a single couple of type } i.$$
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Theorem [Fritsch - Villemonais - Z.]

Assume $R < +\infty$. For all $z \in \mathbb{N}^p$ there exists a positive r.v. $\mathcal{W}$ such that

$$\frac{Z_n}{(\lambda^*)^n} \xrightarrow[n\to+\infty]{} \mathcal{W}z^*,\ \frac{F_n}{(\lambda^*)^{n+1}} \xrightarrow[n\to+\infty]{} \mathcal{W}z^*\mathbb{X},\ \frac{M_n}{(\lambda^*)^{n+1}} \xrightarrow[n\to+\infty]{} \mathcal{W}z^*\mathbb{Y},$$

$\mathbb{P}(\cdot \mid Z_0 = z)-$a.s..
If $\mathbb{E}(X_{i,j}\log X_{i,j}) < +\infty$, $\mathbb{E}(Y_{i,j}\log Y_{i,j}) < +\infty$ for all i,j + some regularity assumptions over ξ and R, then the limit is in L^1 and for all z and up to a $\mathbb{P}(\cdot \mid Z_0 = z)$ negligible event,

$$\{\mathcal{W} = 0\} = \{\exists n \in \mathbb{N},\ Z_n = 0\}.$$

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Examples

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   - $n_f = n_m = p$.
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   - A particular case: $\mathbb{F} = \alpha\mathbb{U}$, $\mathbb{M} = (1-\alpha)\mathbb{U}$.
     In this case $\lambda^* = \min\{\alpha, 1-\alpha\}\lambda^*_{\mathbb{U}}$, $z^* = z^*_{\mathbb{U}}$.

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2. Multi-type completely promiscuous mating [Karlin - Kaplan, 1973]:
   - $p = n_f$.
   - $\xi(x, y) = x \prod_{i=1}^{n_m} \mathbb{1}_{y_i > 0}$.
   - $R(z) = (z\mathbb{F})\mathbb{1}_{z\mathbb{M} > 0}$.
   - In this case $\lambda^* = \lambda^*_{\mathbb{F}}$.