The Multi-Type bisexual Galton-Watson branching process

Nicolás Zalduendo Vidal

Joint work with Coralie Fritsch and Denis Villemonais

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Department of Statistics
Oxford University

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The Galton-Watson process

Definition

Given $Z_0 = z_0$, we define for $n \geq 0$

$$Z_{n+1} = \sum_{k=0}^{Z_n} X_k^{(n)}$$

where $(X_k^{(n)})_{k,n \in \mathbb{N}}$ are i.i.d. random variables.

$Z_0 = 3$

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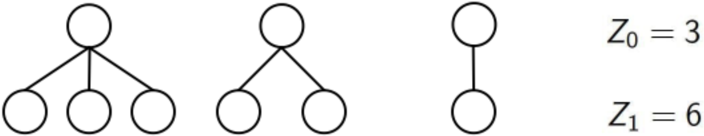

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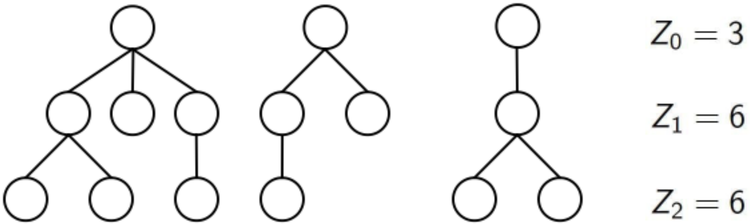

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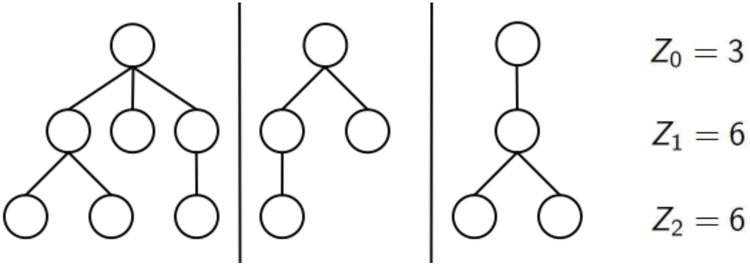

Very important property: INDEPENDENCE!

Extinction Condition

If $\mathbb{P}(Z_1 = 1 | Z_0 = 1) < 1$, then

$$m := \mathbb{E}(Z_1 | Z_0 = 1) \leq 1 \iff Z_n \to 0 \text{ a.s.}$$

The Multi-Type Galton-Watson process

We now consider a process with types:

Definition

Consider $p \in \mathbb{N}$. Given $Z_0 = (z_0^1, \dots, z_0^p)$ we define for $n \geq 0$

$$Z_{n+1}^j = \sum_{i=1}^{p} \sum_{k=1}^{Z_n^\ell} X_{i,j}^{(k,n)}$$

where for all $i, j \in \{1, \dots, p,\}$, $(X_{i,j}^{(k,n)})_{k,n\in\mathbb{N}} \sim_{\text{i.i.d.}} X_{i,j}$.

$Z_0 = (1, 1, 1)$

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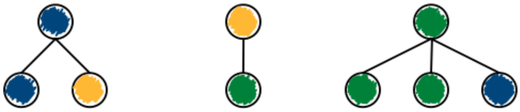

$Z_0 = (1, 1, 1)$

$Z_1 = (2, 1, 3)$

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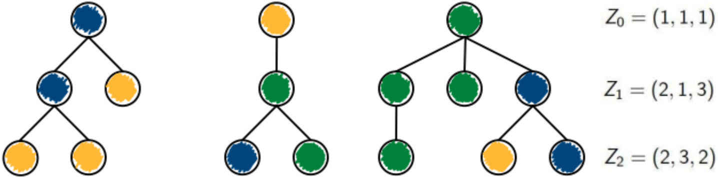


The Multi-Type Galton-Watson process

Define $\mathbb{A}_{i,j} = \mathbb{E}(X_{i,j}) = \mathbb{E}(Z_1^j | Z_0 = e_i)$.

Extinction Condition

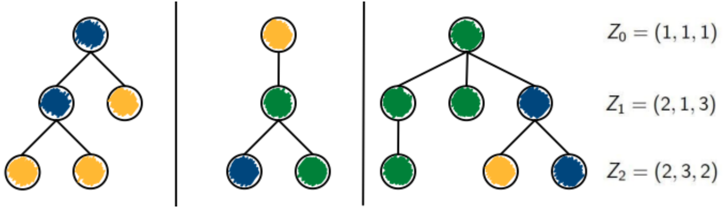
Assume that

- $\mathbb{P}(|Z_1| = 1 || Z_0| = 1) < 1$.
- $\exists N \in \mathbb{N}$, such that $\mathbb{A}^N > 0$.

Then

$$\lambda^* \leq 1 \iff Z_n \to 0, \ a.s.$$

with λ^* the greatest eigenvalue of $\mathbb{A}$.



The bisexual GW branching process [Daley, '68]

Definition

Given $\xi : \mathbb{N}^2 \longrightarrow \mathbb{N}$ with $\xi(0,0) = 0$ and $Z_0 = z_0$, we define for $n \geq 0$,

$$F_{n+1} = \sum_{k=1}^{Z_n} X_k^{(n)}, \; M_{n+1} = \sum_{k=1}^{Z_n} Y_k^{(n)}$$

where $(X_k^{(n)})_{k,n\in\mathbb{N}} \sim_{\text{i.i.d.}} X$ and $(Y_k^{(n)})_{k,n\in\mathbb{N}} \sim_{\text{i.i.d.}} Y$.
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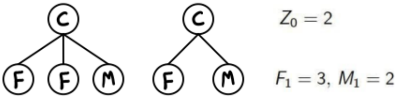

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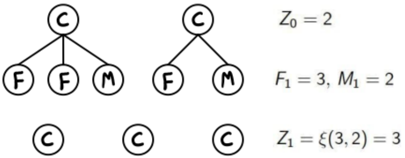

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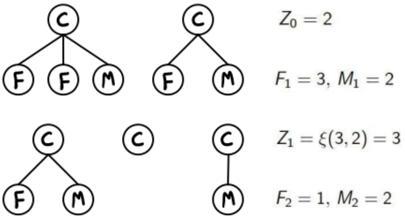

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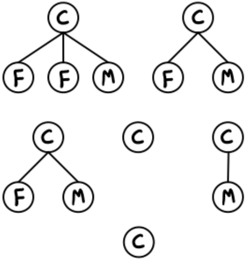


$Z_0 = 2$

$F_1 = 3, \; M_1 = 2$

$Z_1 = \xi(3,2) = 3$

$F_2 = 1, \; M_2 = 2$

$Z_2 = \xi(1,2) = 1$

The bisexual GW branching process [Daley, '68]

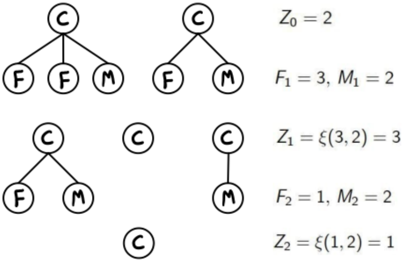
Daley's Mating Functions

- Completely promiscuous mating function ("Cows and Bulls model")

$$\xi(x, y) = x \min\{y, 1\}$$

- Polygamous mating with perfect fidelity

$$\xi(x, y) = \min\{x, dy\}$$



Superadditive Model

Superadditive mating function [Hull, '82]:

$$\xi(x_1 + x_2, y_1 + y_2) \geq \xi(x_1, y_1) + \xi(x_2, y_2),\ \forall x_1, x_2 \in \mathbb{R}_+$$

Implies the existance of:

$$r = \lim_{k \to \infty} \frac{\mathbb{E}(Z_1 | Z_0 = k)}{k}$$

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Theorem [Daley - Hull - Taylor, '86]

$$r \leq 1 \Longleftrightarrow \forall k \in \mathbb{N}, \ \mathbb{P}\left(Z_n \xrightarrow[n \to \infty]{} 0 \middle| Z_0 = k \right) = 1.$$

What about Multi-Type?

Some Multi-Type models that have been studied:

- Mode, 1972: *A 3-type bisexual model where the couple inherits the type of the male.*
- Karlin - Kaplan, 1973: *A Multi-Type version of the Cows and Bulls model, where the couple inherits the type of the female.*
- Hull, 1998: *A 2-type bisexual model where the couple inherits the type of the male.*

But not as deeply as the previous processes!

The Multi-Type bGWbp

Definition

Given $\xi : (\mathbb{R}_+)^{n_f} \times (\mathbb{R}_+)^{n_m} \to (\mathbb{R}_+)^p$ with $\xi(0,0) = 0$ and $Z_0 = (z_0^1, \ldots, z_0^p)$. We define for $n \geq 0$,

$$F_{n+1}^j = \sum_{i=1}^{p} \sum_{k=1}^{Z_n^i} X_{i,j}^{(k,n)}, \text{ for } 1 \leq j \leq n_f, \; M_{n+1}^j = \sum_{i=1}^{p} \sum_{k=1}^{Z_n^i} Y_{i,j}^{(k,n)}, \text{ for } 1 \leq j \leq n_m,$$

where for all i, j, $(X_{i,j}^{(k,n)})_{k,n\in\mathbb{N}} \sim_{\text{i.i.d}} X_{i,j}$ and $(Y_{i,j}^{(k,n)})_{k,n\in\mathbb{N}} \sim_{\text{i.i.d.}} Y_{i,j}$. We set $(Z_{n+1}^1, \ldots, Z_{n+1}^p) = \xi((F_{n+1}^1, \ldots, F_{n+1}^{n_f}), (M_{n+1}^1, \ldots, M_{n+1}^{n_m}))$.

$Z_0 = (1, 2)$

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$F_1 = (1, 1, 2)$, $M_1 = (3, 2)$

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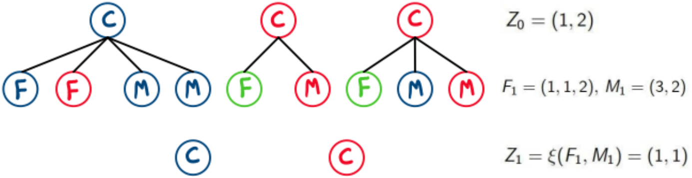

The Multi-Type bGWbp

Assumptions:

- Superadditivity:

$$\xi(x_1+x_2, y_1+y_2) \geq \xi(x_1, y_1) + \xi(x_2, y_2).$$

- Integrability: The matrices

$$\mathbb{F}_{i,j} = \mathbb{E}(X_{i,j}) = \mathbb{E}(F_1^j | Z_0 = e_i),\ \mathbb{M}_{i,j} = \mathbb{E}(Y_{i,j}) = \mathbb{E}(M_1^j | Z_0 = e_i)$$

  are well defined.
- Independence: For $i_1 \neq i_2$

$$X_{i_1,j} \perp X_{i_2,j} \text{ and } Y_{i_1,j} \perp Y_{i_2,j}, \text{ for all } j$$

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Proposition

The function $R : \mathbb{N}^p \longrightarrow (\mathbb{R}_+ \cup \{+\infty\})^p$ given by

$$R(z) = \lim_{k \to +\infty} \frac{\mathbb{E}(Z_1 | Z_0 = kz)}{k}$$

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Law of Large Numbers

What is the role of R?

$$R(z) = \lim_{k \to +\infty} \frac{\mathbb{E}(Z_1 | Z_0 = kz)}{k}.$$

Theorem [Fritsch - Villemonais - Z.]

Assume $R < \infty$ and let $(z_k)_{k \geq 1} \in (\mathbb{N}^p)^{\mathbb{N}}$ be a sequence such that $z_k \sim_{k \to +\infty} kz \in \mathbb{R}_+^p$ a.s., and, for all $k \geq 1$, denote by $(Z_{k,n})_{n \geq 0}$ the bGWbp with $Z_{k,0} = z_k$. Then, for all $n \geq 0$,

$$Z_{k,n} \sim_{k \to +\infty} R^n(kz) \text{ a.s..}$$

If $\sup_{k \geq 1} \frac{z_k}{k}$ is bounded, $Z_{k,n}/k$ converges to $R^n(z)$ in L^1.

Law of Large Numbers

Special Case: $z_k = kz \in \mathbb{N}^p$.

- LLN + ξ superadditive implies

$$\frac{Z_{k,1}}{k} \xrightarrow{k \to +\infty} \lim_{k\to\infty} \frac{\xi(kz\mathbb{F}, kz\mathbb{M})}{k} \text{ a.s.}$$

- $(Z_{k,1}/k)_{k\in\mathbb{N}}$ is U.I, which implies

$$R(z) = \lim_{k\to\infty} \frac{\mathbb{E}(Z_{k,1})}{k} = \lim_{k\to\infty} \frac{\xi(kz\mathbb{F}, kz\mathbb{M})}{k}.$$

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Lemma

For any $z \in \mathbb{N}^p$,

$$R(z) = \lim_{k\to+\infty} \frac{\xi(kz\mathbb{F}, kz\mathbb{M})}{k} = \sup_{k>0} \frac{\xi(kz\mathbb{F}, kz\mathbb{M})}{k}$$

Fact: The function R is **concave**.

Condition for certain extinction

Extra assumptions:

- Transcience:

$$\mathbb{P}(Z_n \to 0 \mid Z_0 = z) + \mathbb{P}(Z_n \to +\infty \mid Z_0 = z) = 1, \quad \forall z \in \mathbb{N}^p \setminus \{0\}.$$

- There exists $N \in \mathbb{N}$ such that for all $n \geq N$,

$$\forall z \in (\mathbb{R}_+)^p,\ R^n(z) > 0$$

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Theorem [Krause, '94]

1. The eigenvalue problem

$$R(z^*) = \lambda^* z^*$$

has a unique solution with $\lambda^* > 0$ and $z^* \in (\mathbb{R}_+)^p$, $z^* > 0$, $|z^*| = 1$.

2. There exists a function $P : (\mathbb{R}_+)^p \longrightarrow \mathbb{R}_+$ such that

$$\lim_{n \to +\infty} \frac{R^n(z)}{(\lambda^*)^n} = P(z) z^*$$

Condition for certain extinction

Define

$$q_z = \mathbb{P}(\exists n \in \mathbb{N},\ Z_n = 0 | Z_0 = z)$$

Condition for certain extinction

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Theorem [Fritsch - Villemonais - Z.]

Assume R is finite. Then,

$$\lambda^* \leq 1 \Longleftrightarrow q_z = 1, \forall z \in \mathbb{N}^p.$$

If $\lambda^* > 1$, then $\forall \varepsilon > 0, \exists v_0 \in \mathbb{N}^p$ such that if $Z_0 = v_0$

$$\mathbb{P}\left(Z_n > (\lambda^* - \varepsilon)^n v_0,\ \forall n \in \mathbb{N}\right) > 0.$$

If there exists $z \in (\mathbb{R}_+)^p$ such that $R(z)$ is not finite, then $q_v < 1$ for some $v \in \mathbb{N}^p$.

Idea of the proof

First assume $R < +\infty$.

- $\underline{\lambda^* > 1}$: Fix $\varepsilon > 0$ such that $\lambda^* - \varepsilon > 1$. Then for $k \in \mathbb{N}$ big enough,

$$\sup_{j>0} \frac{\xi(jz^*\mathbb{F}, jz^*\mathbb{M})}{j} = \lambda^* z^* \Longrightarrow \xi(kz^*\mathbb{F}, kz^*\mathbb{M}) \geq (\lambda^* - \varepsilon)kz^* > kz^*$$

Using this,

$$\mathbb{P}\left(Z_1 > (\lambda^* - \varepsilon)Z_0 | Z_0 = kz^*\right) \geq 1 - \frac{C}{k}, \text{ for some } C > 0.$$

Thanks to the Markov property:

$$\mathbb{P}\left(\bigcap_{n=1}^{+\infty}\{Z_n > (\lambda^* - \varepsilon)^n v_0\} \middle| Z_0 > v_0 = kz^*\right) \geq \prod_{n=0}^{+\infty}\left(1 - \frac{C}{(\lambda^* - \varepsilon)^n}\right) > 0$$

Idea of the proof

- $\underline{\lambda^* \leq 1}$: Since

$$\mathbb{E}(Z_n|Z_0 = z) \leq R^n(z)$$

Krause's second statement $\implies$ $(R^n(z))_{n\in\mathbb{N}}$ is bounded.
Transience $\implies$ extinction.

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Transience $\implies$ extinction.

If R is not finite: Construct superadditive functions ξ_α sucht that as $\alpha \to +\infty$:

- $\xi_\alpha(x,y) \to \xi(x,y)$ for all x and all y.
- Associated functions R_α are all finite and hold $R_\alpha(z) \to R(z)$ for all z.
- The sequence of eigenvalues $\lambda_\alpha \to +\infty$.

Choose $\hat{\alpha}$ with $\lambda_{\hat{\alpha}} > 1$. The associated process is supercritical and stochastically dominates our process from below.

□

The Multi-Type bisexual Galton-Watson process

	Asexual Process	Bisexual Process
Single-Type	*Classic Galton-Watson process*	*Bisexual Galton-Watson process*
	$p = 1,\ \xi(x, y) = x$ $R(z) = mz$ Extinction condition: $R(1) \leq 1$	$p = 1,\ \xi$ superadditive $R(z) = rz$ Extinction condition: $R(1) \leq 1$
Multi-Type	*Multi-Type Galton-Watson process*	
	$p > 1,\ \xi(x, y) = x$ $R(z) = z\mathbb{A}$ Extinction condition: $\lambda^* \leq 1$	

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Multi-Type	*Multi-Type Galton-Watson process*	*Multi-Type bGWbp*
	$p > 1$, $\xi(x, y) = x$ $R(z) = z\mathbb{A}$ Extinction condition: $\lambda^* \leq 1$	$p > 1$, ξ superadditive R concave Extinction condition: $\lambda^* \leq 1$

Examples

Some examples:

1. Multi-Type perfect fidelity mating:
   - $n_f = n_m = p$.
   - $\xi(x, y) = \min\{x, y\}$.
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2. Multi-Type completely promiscuous mating [Karlin - Kaplan, 1973]:
   - $p = n_f$.
   - $\xi(x, y) = x \prod_{i=1}^{n_m} \mathbb{1}_{y_i > 0}$.
   - $R(z) = (z\mathbb{F})\mathbb{1}_{z\mathbb{M} > 0}$.
   - In this case $\lambda^* = \lambda^*_{\mathbb{F}}$.

Asymptotic Behaviour

What can we say about the asymptotic behavior of the process?

Conjecture

There exists a real and positive random variable W such that

$$\frac{Z_n}{(\lambda^*)^n} \xrightarrow[a.s.]{n\to\infty} Wz^*$$

If the conjecture is true, we want to find conditions for:

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Lemma

If $R < +\infty$, $\lambda^* > 1$, + technical assumptions: There exists a finite random variable $\mathcal{P}$ such that

$$\frac{P(Z_n)}{(\lambda^*)^n} \xrightarrow[\text{a.s., } L^1]{n\to+\infty} \mathcal{P}$$

with $\mathbb{E}(\mathcal{P}|Z_0 = z) > 0$ for all $z \in \mathbb{N}$ such that $q_z < 1$.

The Multi-Type bisexual Galton-Watson branching process

Nicolás Zalduendo Vidal

Joint work with Coralie Fritsch and Denis Villemonais

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